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MATH 491-904

A Historical Survey of French Number Theory

1 Introduction

The objective of this paper is both to provide historical context for the development of number theory, and to examine content that might be unavailable to the some readers because of linguistic limitations. It is particularly illuminating to see how the field developed in its infancy, and the ingenious methods employed in the search for intermediate results of what would become with time iconic theorems.

This paper will examine two lines of development concerning the work leading up to the Prime Number Theorem and Fermat's Last Theorem as made by Adrien Legendre, P. L. Chebychev, and Augustin Cauchy. Particularly in the case of the latter, the work we will examine is far removed from the machinery that would eventually be required for a proof of such a theorem, but is interesting in light of the insight to how a masterful mathematician approached a new subject.

2 Legendre Conjecture Concerning the Distribution of Primes

In Adrien-Marie Legendre's textbook Théorie Des Nombres, Legendre makes the first conjectures regarding the number of primes less than a given value. He begins by noting that the formula

$$y = \frac{x}{\log x - 1.08366}$$

appears to approximate the number of primes less than a given value x, and provides the below table for empirical verification.

x	Estimate of y by Formula	Result from Tables
10000	1250	1250
20000	2268	2263
30000	3252	3246
40000	4205	4204
50000	5136	5134
60000	6049	6058
70000	6949	6936
80000	7838	7837
90000	8717	8713
100000	9588	9592
150000	13844	13849
200000	17982	17984
250000	22035	22045
300000	26023	25998
350000	29961	29977
400000	33854	33861

Legendre conjectures that the prime counting function in general is of the form:

$$y = \frac{x}{A \log x + B}$$

with the particular values of the coefficients as above.

Legendre proceeds off the assumption this distribution is correct, and turns to the issue of considering the distance between successive primes, and the number of primes on an arbitrary interval. Letting $c = 1.08366$ and α represent the amount x must be increased to find another prime i.e the number such that:

$$y + 1 = \frac{x + \alpha}{\log(x + \alpha) - c}$$

and thus substituting the definition of our function and rearranging terms:

$$1 = \frac{x + \alpha}{\log(x + \alpha) - c} - \frac{x}{\log x - c}$$

And assuming sufficiently large x such that $\log x \approx \log(x+\alpha)$ this simplifies to:

$$\alpha = (\log x - c + 1)\left(1 + \frac{1}{2x}\right)$$

which taking the limit as x approaches infinity gives:

$$\alpha = \log x - .08366$$

which is the average distance between successive primes. From the current perspective of knowing the prime number theorem gives the above constant B to be 0, we can see that the logarithmic term of this calculation is correct, without any need for the addition of a constant. Legendre then notes that, given his above calculations, on an interval $(x-m, x+m)$ for some arbitrary integers x, m that the number of primes in the interval is equal to:

$$\frac{2m}{\log x - .08366}$$

provided that m is small with respect to x, or in modern terminology, that this approximation becomes exact as x tends toward infinity.

Legendre now turns his attention to the following product, easily recognized as the Riemann zeta function for real integers with $s = 1$, where p ranges over the odd primes:

$$z = \prod_{p=3}^{\omega}\left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\ldots\left(1 - \frac{1}{\omega}\right)$$

and where:

$$z^{'} = z\left(1 - \frac{1}{\omega + \alpha}\right) = z\left(\frac{\omega + \alpha - 1}{\omega + \alpha}\right)$$

Then finding the Taylor series gives:

$$z^{'} = \sum_{n=0}^{\infty}\frac{1}{n!}\alpha^n\frac{d^n z}{d\omega^n} = z + \alpha\frac{dz}{d\omega} + \frac{1}{2}\alpha^2\frac{d^2 z}{d\omega^2} + \ldots$$

Noticing that:

$$\frac{dz}{z} = \frac{-d\omega}{\omega\alpha} = \frac{-d\alpha}{\alpha}$$

looking at only the terms of first order and assuming that α is small with respect to ω yields the following approximation:

$$z = \frac{A}{\alpha} = \frac{A}{\log x - .08366}$$

Or considering terms of the second order as well:

$$z = \frac{A\left(1 - \frac{1}{2\omega}\right)}{\log x - .08366 + \frac{1}{2\omega}}$$

Legendre concludes this section by noting that $A = 1.104$ provides a good approximation that agrees with the above table. Again, we can see how close Legendre was to conjecturing the prime number theorem. What is also of interest is the way the Legendre relates the distribution of prime numbers less than a given value to the partial products that form a sequence (ignoring Legendre's inclusion of erroneous constants) to the Riemann zeta function.

From Legendre's remarks in the conclusion of the article, we can see that he is surprised to find a connection between calculus and number theory, which on the surface do not seem to be intrinsically linked upon a cursory examination, but upon closer examination are evidently some of the most apt tools to investigate the properties of prime numbers.

3 Chebychev's Attempted Proof of the Prime Number Theorem

Theorem I

Let $\varphi(x)$ be the number of primes less than x, and ρ a positive real number. Then provided that ρ converges to zero, the sum:

$$\sum_{x=2}^{\infty}\left[\varphi(x+1) - \varphi(x-1) - \frac{1}{\log(x)}\right]\frac{\log^n x}{x^{\rho+1}} \tag{1}$$

converges to a finite limit.

Proof:

Let m be the index of all integers greater than or equal to 2, and μ the index of all primes greater than or equal to 2. Chebychev shows first, through explicit calculation, that the following sums:

$$\sum \frac{1}{m^{1+\rho}} - \frac{1}{\rho}$$

$$\log \rho - \sum \log \left(1 - \frac{1}{\mu^{1+\rho}}\right)$$

$$\sum \log \left(1 - \frac{1}{\mu^{1+\rho}}\right) + \sum \frac{1}{\mu^{1+\rho}}$$

all satisfy the condition that all their derivatives tend toward finite quantities, provided that ρ converges to zero. Considering that these calculations are made using simple exponential integrals, they need not be repeated here.

Then, Chebychev shows that the expression:

$$\frac{d^n \left[\sum \log \left(1 - \frac{1}{\mu^{1+\rho}}\right) + \sum \frac{1}{\mu^{1+\rho}}\right]}{d\rho^n} - \frac{d^n \left[\log \rho - \sum \log \left(1 - \frac{1}{\mu^{1+\rho}}\right)\right]}{d\rho^n} - \frac{d^{n-1} \left(\sum \frac{1}{m^{1+\rho}} - \frac{1}{\rho}\right)}{d\rho^{n-1}}$$

can be reduced to:

$$\pm \left(\sum \frac{\log^n \mu}{\mu^{1+\rho}} - \sum \frac{\log^{n-1} m}{m^{1+\rho}}\right) \tag{2}$$

Notice that by multiplying and creating two summations our original expression (1) from the statement of the proof can be rewritten as:

$$\sum_{x=2}^{\infty} \left[\varphi(x+1) - \varphi(x-1)\right] \frac{\log^n x}{x^{1+\rho}} - \sum_{x=2}^{\infty} \frac{\log^{n-1} x}{x^{1+\rho}}$$

By looking at the first summation and noticing that $\varphi(x+1) - \varphi(x)$ is equal to 1 for prime numbers and 0 for composite numbers, we see this is essentially equivalent to indexing x for the first summation through the primes as μ, and the second summation through the integers greater than or equal to 2, the same as the index m. Thus we have:

$$\sum_{x=2}^{\infty}\left[\varphi(x+1)-\varphi(x-1)-\frac{1}{\log(x)}\right]\frac{\log^n x}{x^{\rho+1}}=\sum\frac{\log^n \mu}{\mu^{1+\rho}}-\sum\frac{\log^{n-1} m}{m^{1+\rho}}$$

which establishes that, under the condition of ρ converging to zero, the left hand side (1) converges to a finite limit.

Theorem II

$$\int_2^x \frac{dx}{\log x}-\frac{\alpha x}{\log^n x}<\varphi(x)<\int_2^x \frac{dx}{\log x}+\frac{\alpha x}{\log^n x} \tag{3}$$

where α is some arbitrarily small positive real number and n an arbitrarily large positive real number.

Proof:

Chebychev proceeds to prove only the upper bound for $\varphi(x)$, but the manner in which this could be repeated for the lower bound is fairly obvious, and thus omitted. Chebychev first assumes the contrary, and sets a as an integer both greater than e^n and the greatest number which satisfies the inequality (3). Under this assumption $x > a$ and:

$$\varphi(x)\geq\int_2^x \frac{dx}{\log x}+\frac{\alpha x}{\log^n x} \quad , \quad \log x > n$$

or equivalently:

$$\varphi(x)-\int_2^x \frac{dx}{\log x}\geq\frac{\alpha x}{\log^n x} \quad , \quad \frac{n}{\log x}<1$$

Chebychev then notes that:

$$\frac{1}{\log x} \sim \int_x^{x+1} \frac{dx}{\log x}$$

and rewrites (1) as:

$$\sum_{x=2}^{x=s} \left[\varphi(x+1) - \varphi(x-1) - \int_x^{x+1} \frac{dx}{\log x} \right] \frac{\log^n x}{x^{\rho+1}}$$

This is further rewritten as:

$$C + \sum_{x=a+1}^{x=s} \left[\varphi(x+1) - \varphi(x-1) - \int_x^{x+1} \frac{dx}{\log x} \right] \frac{\log^n x}{x^{\rho+1}}$$

where C represents the sum from $x = 2$ through $x = a$. Note that by Theorem I, C is finite given that $\rho \geq 0$. Chebychev then cites the following property of sums:

$$\sum_{a+1}^{s} u_x(v_{x+1} - v_x) = u_s v_{s+1} - u_a v_{a+1} - \sum_{a+1}^{s} v_x(u_x - u_{x-1})$$

and proceeds to take the arguments:

$$v_x = \varphi(x) - \int_2^x \frac{dx}{\log x} \quad , \quad u_x = \frac{\alpha x}{\log^n x}$$

This yields:

$$C - \left[\varphi(a+1) - \int_2^{a+1} \frac{dx}{\log x} \right] \frac{\log^n a}{a^{1+\rho}} + \left[\varphi(s+1) - \int_2^{s+1} \frac{dx}{\log x} \right] \frac{\log^n s}{s^{1+\rho}}$$

$$- \sum_{x=a+1}^{x=s} \left[\varphi(x) - \int_2^x \frac{dx}{\log x} \right] \left[\frac{\log^n x}{x^{1+\rho}} - \frac{\log^n x(x-1)}{(x-1)^{1+\rho}} \right]$$

Chebychev then abbreviates the terms before the summation as F and in the summation substitutes $x - 1$ with $x - \theta$, where $0 < \theta < 1$. After some clever factoring, this gives: Intuitively, this might be viewed as a circuitous method of viewing the convergence of the right handed limit at $x = 0$.

$$F + \sum_{x=a+1}^{x=s} \left[\varphi(x) - \int_2^x \frac{dx}{\log x}\right] \left[1 + \rho - \frac{n}{\log(x-\theta)}\right] \frac{\log^n(x-\theta)}{(x-\theta)^{2+\rho}}$$

From the conditions $\rho > 0$, $x \geq a+1$, and $0 < \theta < 1$, we have:

$$1 + \rho - \frac{n}{\log(x-\theta)} > 1 - \frac{n}{\log a}$$

and from our previous assumption:

$$\varphi(x) - \int_2^x \frac{dx}{\log x} \geq \frac{\alpha x}{\log^n x}$$

combined with the fact that:

$$\frac{d}{dx}\left(\frac{\alpha x}{\log^n x}\right) = \frac{\alpha}{\log^n x}\left(1 - \frac{n}{\log x}\right) > 0 \implies \frac{\alpha x}{\log^n x} > \frac{\alpha(x-\theta)}{\log^n(x-\theta)}$$

Using these inequalities Chebyshev notes that:

$$F + \sum_{x=a+1}^{x=s} \left[\varphi(x) - \int_2^x \frac{dx}{\log x}\right] \left[1 + \rho - \frac{n}{\log(x-\theta)}\right] \frac{\log^n(x-\theta)}{(x-\theta)^{2+\rho}}$$
$$> F + \sum_{x=a+1}^{x=s} \frac{\alpha(x-\theta)}{\log^n(x-\theta)} \left(1 - \frac{n}{\log x}\right) \frac{\log^n(x-\theta)}{(x-\theta)^{2+\rho}}$$

This reduces to:

$$F + \alpha\left(1 - \frac{n}{\log a}\right) \sum_{x=a+1}^{x=s} \frac{1}{(x-\theta)^{1+\rho}} > F + \alpha\left(1 - \frac{n}{\log a}\right) \sum_{x=a+1}^{x=s} \frac{1}{x^{1+\rho}}$$

Taking $s = \infty$ for the right hand side gives:

$$F + \alpha\left(1 - \frac{n}{\log a}\right) \sum_{x=a+1}^{\infty} \frac{1}{x^{1+\rho}} = F + \alpha\left(1 - \frac{n}{\log a}\right) \frac{\int_0^\infty \frac{e^{-ax}}{e^x + 1} x^\rho \, dx}{\int_0^\infty e^{-x} x^\rho \, dx}$$

But taking $\rho = 0$, we have:

$$\int_0^\infty \frac{e^{-ax}}{e^x + 1} \, dx = \infty \quad , \quad \int_0^\infty e^{-x} \, dx = 1$$

This is a contradiction since it implies that:

$$\sum_{x=2}^{\infty} \left[\varphi(x+1) - \varphi(x-1) - \frac{1}{\log(x)} \right] \frac{\log^n x}{x^{\rho+1}} = \infty$$

which contradicts the first theorem. Consequently, our first assumption must have been faulty, proving the bounds:

$$\int_2^x \frac{dx}{\log x} - \frac{\alpha x}{\log^n x} < \varphi(x) < \int_2^x \frac{dx}{\log x} + \frac{\alpha x}{\log^n x}$$

The (Conditional) Prime Number Theorem

If the following limit:

$$\lim_{x \to \infty} \frac{x}{\varphi(x)} - \log x$$

exists, it must be equal to -1.

Note that this limit existing is equivalent to saying that there exist some (large) positive N such that $\dfrac{x}{log^N(x)}$ approximates $\varphi(x)$ as x tends to infinity.

Proof:

Let:

$$\lim_{x \to \infty} \frac{x}{\varphi(x)} - \log x = L$$

By the definition of a limit $\exists N \in \mathbb{R}$ such that $\forall x > N$

$$L - \epsilon < \frac{x}{\varphi(x)} - \log x < L + \epsilon$$

Using the bounds on $\varphi(x)$ found by the previous theorem, this is equivalent to:

$$L - \epsilon < \frac{x}{\displaystyle\int_2^x \frac{dx}{\log x} - \frac{\alpha x}{\log^n x}} - \log x$$

and:

$$\frac{x}{\displaystyle\int_2^x \frac{dx}{\log x} + \frac{\alpha x}{\log^n x}} - \log x < L + \epsilon$$

Simplifying we have:

$$\frac{x - (\log x - 1)\left(\displaystyle\int_2^x \frac{dx}{\log x} + \frac{\alpha x}{\log^n x}\right)}{\displaystyle\int_2^x \frac{dx}{\log x} + \frac{\alpha x}{\log^n x}} - \epsilon < L + 1 < \frac{x - (\log x - 1)\left(\displaystyle\int_2^x \frac{dx}{\log x} - \frac{\alpha x}{\log^n x}\right)}{\displaystyle\int_2^x \frac{dx}{\log x} - \frac{\alpha x}{\log^n x}} + \epsilon$$

Noting that:

$$\lim_{x\to\infty} \frac{x - (\log x - 1)\left(\displaystyle\int_2^x \frac{dx}{\log x} \mp \frac{\alpha x}{\log^n x}\right)}{\displaystyle\int_2^x \frac{dx}{\log x} \mp \frac{\alpha x}{\log^n x}} \pm \epsilon = \pm\epsilon$$

we have:

$$-\epsilon < L + 1 < \epsilon \implies -1 - \epsilon < L < -1 + \epsilon \implies L = -1$$

Intuitively, it seems obvious that this limit must exist, and it seems reasonable to follow this distribution, aligning with the intuitive perception that prime numbers become increasingly scarce. However, putting this intuition aside, more sophisticated methods are required to provide a rigorous proof. In particular, nearly every proof of the theorem relies on machinery from complex analysis.

Unfortunately the complete proof from Charles Jean de la Vallee-Poussin and Jacques Hadamard, the first to independently prove the prime number theorem, both utilize complex analysis and the behavior of the Riemann zeta function in their poofs, and fall beyond the scope of this paper and the ability of its author.

4 Cauchy's Work on Fermat's Last Theorem

While Cauchy obviously did not resolve Fermat's last theorem, he worked quite diligently on the subject, and in his attempts to do so made significant and interesting advances in number theory, both directly and indirectly related to Fermat's Last Theorem. Throughout the 1840s, Cauchy attempted to prove the theorem by analyzing polynomials at roots of unity, and radical polynomials. While presenting the entirety of his work on the matter would be impossible, it is certainly within the scope of this paper to examine a few representative examples.

4.1 A sum related to Fermat's Last Theorem

Cauchy begins by considering ρ, a primitive root of the equation $x^n = 1$, and letting $f(\rho)$ be a polynomial function of ρ. He then suppose that $1 - \rho$ is a factor of $f(\rho)$ and notes that this forces:

$$f(1) \equiv 0 \mod n$$

Now Cauchy supposes that $f(\rho)$ instead has a factor of $(1-\rho)^l$ and writes:

$$f(\rho) = \varphi(\rho)\chi(\rho)$$

where $\varphi(\rho)$ and $\chi(\rho)$ both have factors of $(1-\rho)^h$ and $(1-\rho)^k$ respectively, and $l = h + k$. Cauchy then notes that if

$$f(\rho) \equiv 0 \mod n$$

and

$$\chi(1) \not\equiv 0 \mod n$$

then

$$\varphi(\rho) \equiv 0 \mod n$$

Cauchy then considers

$$X = \frac{x^n - 1}{x - 1}$$

and notes that all of its derivatives through order $m - 2$ evaluated at $x = 1$ are congruent to 0 modulo n. With this in mind, he defines (where ranges through the positive integers and $f_0(x)$ is defined as just $f(x)$)

$$f_m(x) = x f'_{m-1}(x)$$

Then as before takes

$$f(x) = \varphi(x)\chi(x)$$

and notes that the basic calculus gives:

$$f_m(x) = \varphi(x)\chi_m(x) + m\varphi_1(x)\chi_{m-1}(x) + \cdots + \varphi_m(x)\chi(x)$$

Using this, Cauchy determines that if

$$f(\rho) \equiv 0 \mod n$$

then

$$f(1) \equiv f_1(1) \equiv f_2(1) \equiv \cdots \equiv f_{n-2}(1) \equiv 0 \mod n$$

Cauchy then supposes that $f(\rho)$ is such that

$$\varphi(\rho) \equiv \varphi(\rho^{-1}) \mod n$$

and notes that this is satisfied when

$$\varphi_1(1) \equiv \varphi_3(1) \equiv \cdots \equiv \varphi_{n-2}(1) \equiv 0 \mod n$$

though not explicitly explained by Cauchy, this is because taking

$$f(x) = \varphi(\rho^{-1})$$

gives

$$f_m(x) = -1^m \varphi(\rho^{-1})$$

Cauchy then defines $u_l = a + b\rho^l$, where $\gcd(a, b) = 1$, letting l range from 1 to $n-1$ and defining a_l to be the inverse of l modulo n, Cauchy defines the function

$$\varphi(\rho) = \rho^\mu u_{a_1} u_{a_2} \ldots u_{a_{\frac{n-a}{2}}}$$

where μ is an integer. Cauchy then defines the function $F(\rho)$ and chooses μ such that

$$\frac{\varphi(\rho^{-1})}{\varphi(\rho)} = \left[\frac{F(\rho^{-1})}{F(\rho)}\right]^n$$

and from our earlier calculations is able to conclude immediately that

$$\varphi(\rho) \equiv \varphi(\rho^{-1}) \mod n$$

What is interesting about this conclusion is that it places a rather strict condition on a function solely by noticing that it is built from a product of powers chosen based upon their relationship between each other in terms of being multiplicative inverses modulo n.

Continuing on, Cauchy now turns to consider s, a primitive root of

$$x^{n-1} \equiv 1 \mod n$$

Again, Cauchy splits the positive integers through $n-1$ into half, this time defining a_h, a_k, indexed as follows

$$a_1, a_2, \dots a_{\frac{n-1}{2}}$$

such that

$$a_k \equiv \pm s^m a_h$$

and furthermore essentially defines a matrix defined by

$$\alpha_{h,m} = s^m \frac{a_h}{a_k}$$

which takes values ± 1 as follows

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,\frac{n-1}{2}} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,\frac{n-1}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\frac{n-1}{2},1} & \alpha_{\frac{n-1}{2},2} & \cdots & \alpha_{\frac{n-1}{2},\frac{n-1}{2}} \end{bmatrix}$$

Furthermore, Cauchy defines ω as the sum of the diagonal entries of this matrix, up to the sign of the sum. Cauchy notes that the aim of this matrix is to note that an integer exponent v such that the following

$$\left(\rho^{-2v} \frac{u_{-l}}{u_1}\right)^{\omega}$$

reduces to, as in the previous section, a polynomial of degree n of the form

$$\frac{\mathscr{F}(\rho^{-1})}{\mathscr{F}(\rho})$$

and in a similar fashion to earlier gives

$$(\rho^v u_l)^{\omega} \equiv (\rho^{-v} u_{-l})^{\omega} \mod n$$

Again, there is considerable interest in seeing that a function, by virtue of being composed of a primitive roots of a particular modular equation, is constrained significantly based upon pairing the numbers less than the given modulus. Calculating the first few ω for prime modulus, Cauchy appears to be returning the triangular numbers.

Finally, having built the machinery throughout the two articles under examination Cauchy shows that if there are no solutions to the equation

$$a^n + b^n + c^n = 0$$

where p is an odd prime that does not divide the product abc, commonly referred to as the first case of Fermat's Last Theorem, then this is equivalent to the sum

$$1 + 2^{n-4} + 3^{n-4} + \cdots + \left(\frac{n-1}{2}\right)^{n-4}$$

not being divisible by n, which is to say that ω is relatively prime to n. Cauchy ends the paper by describing how in practice to solve the resulting systems of linear equations that result.

Here we can finally see the fundamental connection that is drawn between the factoring of polynomial equations and Fermat's Last Theorem. Looking at the year that these papers were published (1847), it is clear that Cauchy is moving in the exact same direction as his contemporaries, slowing moving toward the idea that extending the ideas of unique factorization from the integers into the complex numbers would be sufficient to prove Fermat's Last Theorem. While this obviously did not occur because in fact factorization need not be unique for all cyclotomic integers, it is undeniable that the motivation of Fermat's Last Theorem pushed mathematicians to expand the boundaries of number theory.

5 Conclusion

Looking at the development of the work leading to the two main theorems examined in this paper, it is obvious that these first attempts were by no means a concentrated effort, or even had behind them the force of some comprehensive program of attack. In the case of Legendre, it is particularly apparent that his work was guided almost solely by his intuition about the distribution of prime numbers, and his ability to conjecture based upon his patient gathering of numerical data. While Chebychev operates with a bit more rigor, the progress he makes towards proving the prime number theorem still holds a cavalier attitude towards such formalities like limits and differentials. This carries with it the freedom to more rapidly, by operating on mathematical intuition about such things as convergence and asymptotic behavior, it often leaves much to be desired in terms of the rigor, and leaps to conclusions that while correct, often require quite a bit of work to be fully fleshed out by modern standards.

Cauchy, despite being remembered today as a contributor to the rigor of mathematics with such ideas as the $\epsilon - \delta$ definition of a limit, still did mathematics with the same kind of informality that would be inappropriate by today's standards for mathematics research. Often it easily seen how much more cleanly and efficiently his work could have been present simply with the addition notation now universally accepted, such as indexing and summation notation.

Despite all of these misgivings about the rigor involved in these works, it is undeniable that these are the papers that shaped the landscape of number theory. Any argument against the value of these papers on the basis of lacking falls flat its face when weighed against the way that these ideas shaped the field of number theory for decades past their writing. If anything, they are more valuable than a more rigorous exposition because of the very fact they rely so heavily on an intuitive understanding of number theory, giving the reader not just knowledge of the propositions and theorems that they provide, but a place to begin whenever venturing on their own into uncharted territory. While these are certainly not the quickest or best paths to the great theorems of number theory, they are certainly the most instructive.

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