Fibonacci Power Series
Author(s): Paul Glaister
Source: *The Mathematical Gazette*, Vol. 79, No. 486 (Nov., 1995), pp. 521-525
Published by: The Mathematical Association
Stable URL: https://www.jstor.org/stable/3618079
Accessed: 14-04-2020 15:33 UTC

Fibonacci power series

PAUL GLAISTER

A student usually first meets power series through an infinite geometric progression, having previously considered finite geometric progressions. In this note we consider a variation of this introductory material which involves the Fibonacci numbers. This necessarily poses various questions, e.g. 'When does the series converge and, if so, what is the sum?'. However, there is one further intriguing question that is natural to ask, and this leads to some interesting mathematics. All of this is appropriate for sixth formers, either for classroom discussion or as an exercise.

To introduce the first series consider the geometric progression

$$\sum_{i=1}^{\infty} t^i = t + t^2 + \dots \qquad (1)$$

which converges to $t/(1 - t)$, provided $-1 < t < 1$. For example, with $t = \frac{1}{2}$, we have

$$\sum_{i=1}^{\infty} (\tfrac{1}{2})^i = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

If we now multiply the terms in (1) by the Fibonacci numbers defined by

$$F_1 = F_2 = 1; \qquad F_i = F_{i-1} + F_{i-2}, \quad i \geqslant 3, \qquad (2)$$

i.e. multiply t^i by F_i, we obtain the series

$$S = \sum_{i=1}^{\infty} F_i t^i \qquad (3)$$

whose convergence is now in question. (The first ten Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34 and 55.) To answer this we attempt to evaluate S by employing definition (2), i.e.

$$\begin{aligned} S = \sum_{i=1}^{\infty} F_i t^i &= F_1 t + F_2 t^2 + \sum_{i=3}^{\infty} F_i t^i \\ &= t + t^2 + \sum_{i=3}^{\infty} (F_{i-1} + F_{i-2}) t^i \\ &= t + t^2 + t \sum_{p=2}^{\infty} F_p t^p + t^2 \sum_{q=1}^{\infty} F_q t^q \\ &= t + t^2 + t(S - t) + t^2 S \end{aligned}$$

and hence

$$S = \sum_{i=1}^{\infty} F_i t^i = \frac{t}{1 - (t + t^2)}. \qquad (4)$$

However we know that the series for $1/(1 - ct)$ converges if and only if $|ct| < 1$, and hence expressing the right hand side of (4) using partial fractions will identify the radius of convergence for the series S.

Factorising the denominator on the right hand side of (4) as $1 - t - t^2 = (1 - at)(1 - bt)$, where $a = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ and $b = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618$, then

$$\frac{t}{1 - t - t^2} = \frac{t}{(1 - at)(1 - bt)} = \frac{A}{1 - at} + \frac{B}{1 - bt},$$

and determination of A and B gives $A = -B = 1/(a - b) = 1/\sqrt{5}$. Hence

$$S = \sum_{i=1}^{\infty} F_i t^i = \frac{1/\sqrt{5}}{1 - at} - \frac{1/\sqrt{5}}{1 - bt}, \tag{5}$$

where a and b are given above. Now, the series for the first term on the right hand side of (5) converges if and only if $|at| < 1$, and similarly the series for the second term on the right hand side converges if and only if $|bt| < 1$. Thus the series S converges if and only if $|t| < |\min(|a^{-1}|, |b^{-1}|)|$.

However, since $0 < -b < a$, with $ab = -1$, then $\min(|a^{-1}|, |b^{-1}|) = -b$ is the radius of convergence, and the range of values for t for which the series S converges is $b < t < -b$, or

$$\tfrac{1}{2}(1 - \sqrt{5}) < t < \tfrac{1}{2}(\sqrt{5} - 1), \tag{6}$$

i.e. $-0.618 < t < 0.618$ approximately. We note that expanding the series on the right hand side of (5) and comparing coefficients of t^i yields the familiar explicit formula for the ith Fibonacci number as

$$F_i = \frac{(a^i - b^i)}{\sqrt{5}}, \tag{7}$$

where a and b are given above, and we make use of this shortly. Alternatively, one could use the recurrence relation (2) and the theory of difference equations to derive (7), and then employ this in (3) to prove (4). This approach is not quite so straightforward, however, when considering generalisations of (3), including those we consider later on.

We now pose the question 'for what values of t does the series (4) converge to an *integer*?'. (Clearly for all rational t in the interval of convergence the sum S is rational.) The corresponding question for the geometric series (1) is straightforward since if the sum $t/(1 - t) = m$ is an integer, then $t = m/(1 + m)$ and we note that any such t is a rational. For example, if $m = 2$, then $t = 2/3$ and $\sum_{i=1}^{\infty} (2/3)^i = 2$. For the series (4), however, the answer is not obvious. To obtain a sum which is 1, say, then $t/(1 - (1 + t^2)) = 1$, and hence $t = -1 \pm \sqrt{2}$. The negative root lies outside the interval of convergence, so only with $t = \sqrt{2} - 1$ will the series converge to 1. It is not difficult to obtain a formula for the required value of t for which the series converges to a prescribed integer, since it is merely the solution of a quadratic equation. What is more taxing, however, is to determine the *rational* values of t for which the sum is an integer value, and it is this more specific problem that we now turn to.

Thus, suppose that the sum S in (4) is the integer $k \geqslant 1$, so that

$$\frac{t}{1 - t - t^2} = k.$$

Rearranging gives the quadratic equation

$$kt^2 + (k+1)t - k = 0,$$

whose roots are

$$t = \frac{-(k+1) \pm \sqrt{(k+1)^2 + (2k)^2}}{2k} \tag{8}$$

which means that there are two possible values of t for the series to converge to k, depending on whether they are in the interval of convergence in (6). We require t to be rational and hence that the discriminant $(k+1)^2 + (2k)^2$ is a perfect square. This can be achieved through Pythagorean triples, i.e. set

$$k + 1 = m^2 - n^2 \qquad \text{and} \qquad 2k = 2mn \tag{9}$$

for some integers $m > n \geqslant 1$. In this case,

$$(k+1)^2 + (2k)^2 = \left(m^2 - n^2\right)^2 + (2mn)^2 = \left(m^2 + n^2\right)^2 \tag{10}$$

and, substituting from (9) and (10) into (8),

$$t = \frac{-(k+1) \pm (m^2 + n^2)}{2k} = \frac{-(m^2 - n^2) \pm (m^2 + n^2)}{2mn}$$

$$= \frac{n}{m} \quad \text{or} \quad \frac{-m}{n}$$

giving the required rational values of t. However, since the product of these values $t_1 t_2 = -1$ and the product of the end points of the interval of convergence in (6) is $-b^2 \in (-1, 0)$, there can be no more than one value of t in the interval of convergence.

To complete the solution, therefore, we require integers m and n satisfying (9). If they can be found then eliminating k from the expressions in (9) yields

$$m^2 - n^2 = mn + 1$$

and 'completing the square' gives

$$(m - \tfrac{1}{2}n)^2 = m^2 - mn + \tfrac{1}{4}n^2 = n^2 + 1 + \tfrac{1}{4}n^2 = 1 + \tfrac{5}{4}n^2$$

i.e. $\qquad (2m - n)^2 = 4 + 5n^2.$

Thus $4 + 5n^2$ must be a perfect square, say $4 + 5n^2 = p^2$, where p is a positive integer. This now completes the solution. The first three cases are as follows:

$n = 1$: $4 + 5n^2 = 9 = 3^2$, so $2m - n = p = 3$, i.e. $2m - 1 = 3$ and hence $m = 2$. Thus $k = mn = 2$ is the sum. The two possible values of t are $n/m = \frac{1}{2}$ and $-m/n = -2$. The latter is outside the interval of convergence and hence $\sum_{i=1}^{\infty} F_i \left(\frac{1}{2}\right)^i = 2$.

$n = 3$: $4 + 5n^2 = 49 = 7^2$, so $2m - n = p = 7$, i.e. $2m - 3 = 7$ and hence $m = 5$. Thus $k = mn = 15$ is the sum. The two possible values of t are $n/m = 3/5$ and $-m/n = -5/3$. The latter is outside the interval of convergence and hence $\sum_{i=1}^{\infty} F_i \left(\frac{3}{5}\right)^2 = 15$.

$n = 8$: $4 + 5n^2 = 324 = 18^2$, so $2m - n = p = 18$. i.e. $2m - 8 = 18$ and hence $m = 13$. Thus $k = mn = 104$ is the sum. The two possible values of t are $n/m = 8/13$ and $-m/n = -13/8$. The latter is outside the interval of convergence and hence $\sum_{i=1}^{\infty} F_i \left(\frac{8}{13}\right) = 104$.

(We note that the corresponding negative values of n yield only values of t

which are outside the interval of convergence. For example, with $n = -1$ then $m = 1$ and both $n/m = -1$ and $-m/n = 1$ are outside the interval of convergence in (6). Similarly for $n = -3, -8$, etc.)

Clearly the next obvious choice for n is 13 since we observe that the values of n appear to be the Fibonacci numbers with even indices, and the corresponding value of m is then the next Fibonacci number. Thus, with $n = F_{2j}$, $j \geqslant 1$, then $m = F_{2j+1}$. One way of proving this is to employ (7), as follows, noting that $ab = -1$.

Suppose $n = F_{2j}$, $j \geqslant 1$, then

$$\begin{aligned} 4 + 5n^2 &= 4 + 5F_{2j}^{\ 2} = 4 + 5\left(\frac{(a^{2j} - b^{2j})^2}{5}\right) \\ &= 4 + a^{4j} - 2(ab)^{2j} + b^{4j} = a^{4j} + 2 + b^{4j} \\ &= a^{4j} + 2(ab)^{2j} + b^{4j} \\ &= \left(a^{2j} + b^{2j}\right)^2. \end{aligned}$$

Thus $4 + 5n^2$ is a perfect square, i.e. $2m - n = p = a^{2j} + b^{2j}$, and hence

$$\begin{aligned} m &= \tfrac{1}{2}(p + n) = \tfrac{1}{2}(a^{2j} + b^{2j} + F_{2j}) \\ &= \frac{1}{2}\left(a^{2j} + b^{2j} + (a^{2j} - b^{2j})/\sqrt{5}\right) \\ &= \left(\frac{1}{2}(1 + \sqrt{5})a^{2j} - \frac{1}{2}(1 - \sqrt{5})\,b^{2j}\right)/\sqrt{5} \\ &= (aa^{2j} - bb^{2j})/\sqrt{5} = (a^{2j+1} - b^{2j+1})/\sqrt{5} \\ &= F_{2j+1} \end{aligned}$$

as required. The relevant value of t is then $n/m = F_{2j}/F_{2j+1}$, which is in the interval of convergence since it is well known that the sequence $F_{2j}/F_{2j+1} \to -b$ as $j \to \infty$ is monotonically increasing, and hence that $0 < F_{2j}/F_{2j+1} < -b$ for all $j \geqslant 1$. The value of S is $k = mn = F_{2j}F_{2j+1}$, so that

$$\sum_{i=1}^{\infty} F_i (F_{2j}/F_{2j+1})^i = F_{2j}F_{2j+1}, \qquad j \geqslant 1. \tag{11}$$

(Note that $t = -m/n$ is outside the interval of convergence.) Readers may like to check that with the negative value of $n = -F_{2j}$, $j \geqslant 1$, then the corresponding value of m is F_{2j-1}, and both n/m and $-m/n$ lie outside the interval of convergence since the sequence $F_{2j-1}/F_{2j} > -b$ for all $j \geqslant 1$, and that $F_{2j}/F_{2j-1} \geqslant 1 > -b$ for all $j \geqslant 1$.

We now turn to some extensions of the series (3), the first one of which is

$$T = \sum_{i=1}^{\infty} i\,F_i\,t^i,$$

whose sum is easily determined by differentiating (4) in the same way that the geometric series can be differentiated to give

$$\frac{d}{dt}\left(1 + t + t^2 + \dots\right) = \frac{d}{dt}\frac{1}{(1 - t)},$$

i.e.
$$1 + 2t + 3t^2 + \dots = \frac{1}{(1 - t)^2}.$$

Thus differentiating (4) and multiplying by t yields

$$t\frac{d}{dt}\left(F_1 t + F_2 t^2 + \dots\right) = t\frac{d}{dt}\frac{t}{\left(1 - (t + t^2)\right)},$$

i.e.
$$T = F_1 t + 2F_2 t^2 + 3F_3 t^3 + \dots = \frac{t\left(t^2 + 1\right)}{\left(1 - (t + t^2)\right)^2}.$$

By factorising $(1 - t - t^2)^2 = (1 - at)^2(1 - bt)^2$ as before, and expressing T in terms of partial fractions as

$$\frac{A'}{(1 - at)} + \frac{B'}{(1 - bt)} + \frac{C'}{(1 - at)^2} + \frac{D'}{(1 - bt)^2}$$

for suitable constants A', B', C' and D', it can be shown that T converges for the same values of t that S does. Readers are left to investigate what rational values of t makes T integral, as well as the extension to $\sum_{i=1}^{\infty} i^2 F_i t^i$, etc.

Finally, two related series that are worth considering are

$$U = \sum_{i=1}^{\infty} F_i^2 t^i,$$

$$V = \sum_{i=1}^{\infty} F_i F_{i+1} t^i.$$

To evaluate these we can use the previous approach and consider U and employ the relation (2). This necessarily introduces the second series V, and readers may like to show that

$$(1 - t - t^2)U = t + 2t^2 V, \tag{12}$$

i.e. U and V are related. To find another relation between U and V it is necessary to consider V and employ (2), and this gives

$$(1 - t)V = U. \tag{13}$$

Combining (12) and (13) then gives the individual sums

$$U = \frac{t(1 - t)}{(1 + t)(t^2 - 3t + 1)},$$

$$V = \frac{t}{(1 + t)(t^2 - 3t + 1)}.$$

We leave readers to examine the question of convergence of these series (by observing that $t^2 - 3t + 1 = (t - a^2)(t - b^2)$ and employing partial fractions), together with the problem of determining rational t for which U (or V) is integral.

PAUL GLAISTER

Department of Mathematics, PO Box 220, University of Reading RG6 2AX