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Generalized Summation Methods for Divergent Sums

1 Classical Convergence

Consider the infinite series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \dots$$

The partial sum of a series is defined as

$$s_n = \sum_{n=0}^{n} a_n = a_0 + a_1 + \dots + a_n$$

and if

$$s = \lim_{n \to \infty} s_n$$

for some finite s, we say that the series converges to s. If a series does not converge, it is referred to as being divergent.

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

which by definition converges to

$$\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \left(\frac{1}{2}\right)^n\right) = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}$$

taking the limit gives that the series converges to 2.

2 Unrigorous Intuition

Consider the following alternative argument for summing the same series from the previous example. Let

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Then multiplying term by term (a potentially unjustified operation) gives

$$2s = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

But this is the same as

$$2s = 2 + s$$

which gives the same value of 2 from our formal definition of convergence. The natural question to then ask is if this method can be applied to sums that fail to converge, assigning some meaningful value to divergent sums.

For instance, consider

$$s = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

This series does not converge because its partial sums oscillate between 1 and 0. However by adding parentheses and factoring, we obtain

$$s = 1 - (1 - 1 + 1 - 1 + \dots)$$

which is the same as saying

$$s = 1 - s$$

which seems to suggest that

$$1 - 1 + 1 - 1 + 1 - 1 + \dots = \frac{1}{2}$$

Similarly, the following

$$s = 1 - 2 + 3 - 4 + 5 - 6 + \dots$$

has partial sums that oscillate without bound, showing that the sum is divergent. Nevertheless, the following demonstration seems to suggest that this sum has a finite value of $\frac{1}{4}$.

$$
\begin{aligned}
s &= 1 - 2 + 3 - 4 + 5 - 6 + \dots \\
s &= \quad\;\; 1 - 2 + 3 - 4 + 5 - 6 + \dots \\
s &= \quad\;\; 1 - 2 + 3 - 4 + 5 - 6 + \dots \\
s &= \quad\quad\quad 1 - 2 + 3 - 4 + 5 - 6 + \dots \\
4s &= 1
\end{aligned}
$$

seems to suggest that this sum has a finite value of $\frac{1}{4}$.

The question is if there is a useful and consistent method by which these intuitive calculations may be made rigorous by defining a new method of summation. First, a careful examination of what properties we would like such a method to have is in order.

3 Tentative Summation Axioms

Regularity The summation method is consistent with the classic notion of summation for convergent sums.

Linearity For two sequences a_n, b_n and arbitrary constant k (and arbitrary bounds)

$$k\left(\sum a_n + \sum b_n\right) = \sum k a_n + \sum k b_n$$

Stability

$$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$$

With these ideas in mind we can now introduce a specific summation method that agrees with the intuitive results found above.

4 Hölder Summation

Given a series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \dots$$

Define

$$H_n^0 = s_n$$

and further define

$$H_n^1 = \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

Note that this is precisely the arithmetic mean of the partial sums of the given series. In general we define

$$H_n^{r+1} = \frac{H_0^r + H_1^r + \cdots + H_n^r}{n+1}$$

and if

$$s = \lim_{n\to\infty} H_n^k$$

we say that the series is summable (H, k) to s, which can be written

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots = s \quad (H, k)$$

In words, the (H, k) method is a representation of how many times the mean of the partial sums of a series must be calculated before it converges to a finite value. Two examples may shed light on the method.

Consider the series encountered earlier

$$\sum_{n=0}^{\infty} (-1^n) = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

As previously mentioned the partial sums are

$$s_0 = 1, \quad s_1 = 0, \quad s_2 = 1, \ldots$$

so taking means of the partial sums we have

$$H_0^1 = 1, \quad H_1^1 = \frac{1}{2}, \quad H_2^1 = \frac{2}{3}, \quad H_3^1 = \frac{1}{2}, \quad H_4^1 = \frac{3}{5}, \quad H_5^1 = \frac{1}{2}, \quad \ldots$$

and since we have

$$\lim_{n\to\infty} H_n^1 = \frac{1}{2}$$

We have that

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2} \quad (H, 1)$$

Now consider the series

$$1 - 2 + 3 - 4 + 5 - \dots$$

we have that

$$s_0 = 1, \quad s_1 = -1, \quad s_2 = 2, \quad s_3 = -2, \quad \dots$$

and taking the means

$$H_0^1 = 1, \quad H_1^1 = 0, \quad H_2^1 = \frac{2}{3}, \quad H_3^1 = 0, \quad H_4^1 = \frac{3}{5}, \quad H_5^1 = 0, \quad \dots$$

taking the means again we have

$$H_0^2 = 1, \quad H_1^2 = \frac{1}{2}, \quad H_2^2 = \frac{5}{9}, \quad H_3^2 = \frac{17}{30}, \quad H_4^2 = \frac{17}{45}, \quad , \dots.$$

and taking the limit we see that

$$1 - 2 + 3 - 4 + 5 - \dots = \frac{1}{4} \quad (H, 2)$$

Interestingly, both of these summations match the intuitive calculations made earlier. Furthermore, because only the process of taking arithmetic means is used, all of our desired axioms for a summation method are met. However, this method is not strong enough to provide a value for a sum that fails to diverge by unbounded growth, rather than oscillation.

5 Intuition from Ramanujan

Consider the series

$$1 + 2 + 3 + 4 + \dots$$

The series is divergent since the partial sums grow without bound, and is for the same reason not (H, k) summable for any k. Despite this, the great number theorist Ramanujan argued that since

$$s = 1 + 2 + 3 + 4 + 5 + \dots$$

$$4s = \qquad 4 + \qquad 8 + \dots$$

Then subtracting the second equation from the first gives

$$-3s = 1 - 2 + 3 - 4 + \dots$$

but under Hölder summation the right hand side sums, giving

$$-3s = \frac{1}{4}$$

which suggests that

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}$$

Again, some further definitions are required to bring rigor to this intuition.

6 Zeta Function Regularization

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

which converges for $\Re(s) > 1$. Zeta regularization is to take as our definition that

$$\sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So by definition, we have

$$\zeta(-1) = 1 + 2 + 3 + 4 + \dots$$

To actually find this value, we can find the analytic continuation of the zeta function as follows. First look at the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s}$$

Rewriting in terms of the zeta function we have

$$\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s)$$

But expanding term by term we also have

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) - \left(\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \dots\right) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \dots$$

which is the Dirichlet eta function, notated as $\eta(s)$

Equating the two, we have

$$\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s) = \eta(s)$$

or

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$

Which gives the desired

$$\zeta(-1) = \frac{\eta(-1)}{-3} = -\frac{1}{12}$$

This method gives rise to further equations such as

$$\zeta(0) = 1 + 1 + 1 + 1 + \dots = -\frac{1}{2}$$

$$\zeta(-2) = 1 + 2^2 + 3^2 + 4^2 + \dots = 1 + 4 + 9 + 16 + \dots = 0$$

$$\zeta(-3) = 1 + 2^2 + 3^2 + 4^2 + \dots = 1 + 8 + 27 + 64 + \dots = \frac{1}{120}$$

7 Application to the Casmir Effect

In particular, zeta function regularization appears in the calculation of the Casmir effect. The central idea is that if two perfectly conductive uncharged plates are placed in a vacuum a few nanometers apart, the plates affect virtual photons and create a net force. While a complete examination of this phenomenon is outside the scope of these notes, it suffices to note that zeta regularization is crucial in the calculation of the vacuum expectation value of the energy of the electromagnetic field

$$\langle E \rangle = \frac{1}{2}\sum_n E_n$$

where n ranges over all possible values to account for standing waves. Obviously, this sum is divergent, but by regularizing the Riemann zeta function, it reduces (after much calculation) to

$$\frac{\langle E \rangle}{A} = -\frac{\pi^2}{6r^3}\zeta(-3) = -\frac{\pi^2}{720r^3}$$

where A is the area of the plates and r the distance between them. Rewriting in terms of Planck's constant and the speed of light, we can find the force per unit area is

$$\frac{F_{Cas}}{A} = -\frac{d}{dr}\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{240r^4}$$

which has been verified experimentally within 15% of the predicted value, a remarkable and practical application of summation methods for divergent sums. This effect, being significant on the scale of nanometers, is an important consideration in microelectromechanical systems (MEMS) and nanotechnology.

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